

On the Rate of Convergence in Limit Theorems for Negative–Binomial Random Sums

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Abstract—The main purpose of this paper is to study the rate of convergence in limit theorems for normalized negative–binomial random sums of independent, identically distributed random variables. The obtained limit distributions correspond to a so-called class of Negative–Binomial Infinitely Divisible distributions. Zolotarev’s probability metric is used as a main research approach in this paper.

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1. INTRODUCTION

Let $\{X, X_j, j \geq 1\}$ be a sequence of independent, identically distributed (i.i.d.) random variables with common mean $\mu_X = \mathbb{E}(X)$ and variance $\mathbb{D}(X) = \sigma_X^2 \in (0, +\infty)$. Let $N_{r,p}$ be a negative-binomial distributed random variable with two parameters $r \in \mathbb{N}$ and $p \in (0, 1)$, having probability mass function

$$\mathbb{P}(N_{r,p} = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \dots$$

Further, assume that $N_{r,p}$ is independent of all $X_j, j \geq 1$. Write

$$S_{N_{r,p}} = X_1 + X_2 + \dots + X_{N_{r,p}}. \quad (1)$$

Following the notation used in [22], the random summation defined in (1) is said to be *negative-binomial random sum*. It is worth pointing out that when $r \equiv 1$, from (1) it follows that the sum

$$S_{N_{1,p}} = X_1 + X_2 + \dots + X_{N_{1,p}}, \quad (2)$$

is called *geometric random sum* (see for instance [6], page 4). Here and from now on, assume that $N_{1,p}$ is independent of all $X_j, j \geq 1$. Hence, the studies of negative-binomial random sums are extensions and generalizations for ones of geometric random sums. Moreover, it is well known that the negative-binomial random sums are interesting research objects with wide areas of applications in telecommunications, network analysis and population genetics, insurance mathematics, etc. (see [29, 22, 23, 21], and the references given there). Especially, since the appearance of concepts on Geometrically Infinitely Divisible (GID) and Geometrically Strictly Stable (GSS), introduced by Klebanov et al. [7], geometric random sums have attracted much attention. A number of results related to geometric random sums and negative-binomial random sums with their applications have been investigated by Kruglov and Korolev [16], Gnedenko and Korolev [16], Gnedenko and Korolev [3], Kalashnikov [6], Sandhya and Pillai Pillai [20], Kozubowski and Podgorski [14], Kozubowski [15], Kotz et al. [9], Bon [2], Grandell [4], Klebanov

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[8], Hung [26], Vellaisamy and Upadhye [29], Korolev et al. [10] and [11], Korolev and Dorofeeva [12], Korolev and Zeifman [13], Sunklodas [22] and [23], Sheeja and Kumar [21], Giang and Hung [17], Hung and Hau [27], Hung [25], Hung and Kien [26], [28], etc.

The mathematical tools have been used in studies of the negative-binomial random sums including Stein's method, Kerstan's method and method of exponents (see [29]), etc. Recently, several results related to normal approximation of negative-binomial random sums were established by Manou-Abi [18] and by Sunklodas in [22, 23]. However, these results were considered when parameter $r \rightarrow \infty$ (see [22] and [23] for more details).

The aim of this paper is to establish the rate of convergence in limit theorems for normalized negative-binomial random sums of i.i.d. random variables, using the probability metric originated by Zolotarev in [30, 31] and [32]. The considered distributions in this paper are belonging to a so-called class of negative-binomial infinitely divisible (NBID) distributions. The class mentioned in this paper is an extension of the class introduced by Klebanov et al. in [7]. The known Rényi's Theorem (1957) in [6] and several limit theorems for geometric random sums are considered as direct consequences of desired theorems in this paper.

The paper is organized as follows. Definition and main properties of Zolotarev's probability metric are recalled in the next section. Section 3 is devoted to present the class of NBID distributions and some special NBID distributions. The rates of convergence for negative-binomial random sums via Zolotarev's probability metric are established in Section 4. Based on obtained results, some concluding remarks are deduced in last section.

In our paper, notations $\stackrel{D}{=}$ and $\stackrel{D}{\rightarrow}$ denote equality and convergence in distribution, respectively.

2. ZOLOTAREV'S PROBABILITY METRIC

Let us denote by $\mathfrak{X}$ the set of random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and by $C_B(\mathbb{R})$ the set of all real-valued, bounded, uniformly continuous functions defined on $\mathbb{R} := (-\infty, +\infty)$, with norm $\|f\| = \sup_{w \in \mathbb{R}} |f(w)|$. Furthermore, for $m \in \mathbb{N} := \{1, 2, \dots\}$, $m < s \leq m+1$ and $\beta = s - m$. Write

$$C_B^m(\mathbb{R}) = \left\{ f \in C_B(\mathbb{R}) : f^{(k)} \in C_B(\mathbb{R}), 1 \leq k \leq m \right\},$$

where $f^{(k)}$ is a derivative function of order $k \in \mathbb{N}$. Moreover, we denote

$$\mathfrak{D}_s = \left\{ f \in C_B^m(\mathbb{R}) : |f^{(m)}(x) - f^{(m)}(y)| \leq |x - y|^\beta \right\}.$$

Definition 1. ([30, 31] and [32]). Let $X, Y \in \mathfrak{X}$. Zolotarev's probability metric of order s on $\mathfrak{X}$ between X and Y , is defined as follows

$$\zeta_s(X, Y; \mathfrak{D}_s) = \sup_{f \in \mathfrak{D}_s} \left| \mathbb{E}[f(X) - f(Y)] \right|.$$

Remark 2.1 ([1, 6, 18, 30, 31 and [32]).

1. Zolotarev's probability metric ζ_s is an ideal metric of order s , i.e., for any $c \neq 0$, and for random variables $X, Y, Z \in \mathfrak{X}$, we have

$$\zeta_s(X + Z, Y + Z; \mathfrak{D}_s) \leq \zeta_s(X, Y; \mathfrak{D}_s),$$

and

$$\zeta_s(cX, cY; \mathfrak{D}_s) = |c|^s \zeta_s(X, Y; \mathfrak{D}_s),$$

where Z is independent of X and Y .

2. Let $\{X_j, j \geq 1\}$ and $\{Y_j, j \geq 1\}$ be two sequences of i.i.d. random variables. Assume that the random variables X_j and Y_j are independent for all $j \geq 1$. Then inequality

$$\zeta_s \left(\sum_{j=1}^n X_j, \sum_{j=1}^n Y_j; \mathfrak{D}_s \right) \leq n \cdot \zeta_s(X_1, Y_1; \mathfrak{D}_s). \quad (3)$$

holds.

3. Let us consider

$$\mathfrak{F}_s = \left\{ f \in \mathfrak{D}_s : |f(x) - f(y)| \leq |x - y| \right\} \cap \{f : \|f\| \leq 1\},$$

and define

$$\zeta_s(X, Y; \mathfrak{F}_s) = \sup_{f \in \mathfrak{F}_s} \left| \mathbb{E}[f(X) - f(Y)] \right|.$$

Then $\zeta_s(X_n, X_0; \mathfrak{F}_s) \rightarrow 0$ as $n \rightarrow \infty$ if and only if $X_n \xrightarrow{D} X_0$ as $n \rightarrow \infty$ (see [6], pages 32–35).

We shall need in the sequel several interesting cases concerned with $s = 2$ and $s = 3$ as follows

- Let $m = 1$ and $s = 2$. Then Zolotarev's probability metric of order 2 is defined by

$$\zeta_2(X, Y; \mathfrak{F}_2) = \sup_{f \in \mathfrak{F}_2} \left| \mathbb{E}[f(X) - f(Y)] \right|,$$

where $X, Y \in \mathfrak{X}$ and

$$\begin{aligned} \mathfrak{D}_2 &= \left\{ f \in C_B^1(\mathbb{R}) : |f'(x) - f'(y)| \leq |x - y| \right\}; \\ \mathfrak{F}_2 &= \left\{ f \in \mathfrak{D}_2 : |f(x) - f(y)| \leq |x - y| \right\} \cap \{f : \|f\| \leq 1\}. \end{aligned}$$

- Let $m = 2$ and $s = 3$. Then Zolotarev's probability metric of order 3 is given as follows

$$\zeta_3(X, Y; \mathfrak{F}_3) = \sup_{f \in \mathfrak{F}_3} \left| \mathbb{E}[f(X) - f(Y)] \right|,$$

where $X, Y \in \mathfrak{X}$ and

$$\begin{aligned} \mathfrak{D}_3 &= \left\{ f \in C_B^2(\mathbb{R}) : |f''(x) - f''(y)| \leq |x - y| \right\}; \\ \mathfrak{F}_3 &= \left\{ f \in \mathfrak{D}_3 : |f(x) - f(y)| \leq |x - y| \right\} \cap \{f : \|f\| \leq 1\}. \end{aligned}$$

The following lemma states one of the most important properties of Zolotarev's probability metric.

Lemma 1. Let $X, Y \in \mathfrak{X}$ with $\mathbb{E}|X|^k < +\infty$ and $\mathbb{E}|Y|^k < +\infty$ for $1 \leq k \leq m$. Then $\zeta_s(X, Y; \mathfrak{F}_s)$ is finite and

$$\zeta_s(X, Y; \mathfrak{F}_s) \leq \sum_{k=1}^m \frac{M_k}{k!} |\mathbb{E}(X^k) - \mathbb{E}(Y^k)| + \frac{1}{m!} (\mathbb{E}|X|^s + \mathbb{E}|Y|^s), \quad (4)$$

where $\|f^{(k)}\| = \sup_{w \in \mathbb{R}} |f^{(k)}(w)|$ and $M_k = \sup_{f \in \mathfrak{F}_s} \|f^{(k)}\|$, and for $m < s \leq m + 1$.

Proof. For any $x \in \mathbb{R}$, according to Taylor series expansion for function $f \in \mathfrak{F}_s$ with Lagrange remainder, we have

$$f(x) = f(0) + \sum_{k=1}^m \frac{f^{(k)}(0)}{k!} x^k + \frac{x^m}{m!} [f^{(m)}(\theta x) - f^{(m)}(0)],$$

where $0 < \theta < 1$. Hence, for all $x, y \in \mathbb{R}$ and $f \in \mathfrak{F}_s$, one has

$$f(x) - f(y) \leq \sum_{k=1}^m \frac{|f^{(k)}(0)|}{k!} (x^k - y^k) + \frac{1}{m!} (|x|^s + |y|^s).$$

On the other hand, since $f \in \mathfrak{F}_s$, it follows that

$$|f^{(k)}(0)| \leq \sup_{w \in \mathbb{R}} |f^{(k)}(w)| = \|f^{(k)}\| \quad \text{for } 1 \leq k \leq m.$$

Therefore,

$$f(x) - f(y) \leq \sum_{k=1}^m \frac{\|f^{(k)}\|}{k!} (x^k - y^k) + \frac{1}{m!} (|x|^s + |y|^s), \quad (5)$$

for $m < s \leq m + 1$.

From inequality (5), it may be concluded that

$$\zeta_s(X, Y; \mathfrak{F}_s) \leq \sup_{f \in \mathfrak{F}_s} \left\{ \sum_{k=1}^m \frac{\|f^{(k)}\|}{k!} |\mathbb{E}(X^k) - \mathbb{E}(Y^k)| + \frac{1}{m!} (\mathbb{E}|X|^s + \mathbb{E}|Y|^s) \right\}.$$

Setting $M_k = \sup_{f \in \mathfrak{F}_s} \|f^{(k)}\|$, the proof is straightforward. $\square$

3. NEGATIVE-BINOMIAL INFINITELY DIVISIBLE DISTRIBUTIONS

We first introduce the notion of a negative-binomial infinitely divisible random variables as follows.

Definition 2. A random variable Y is said to be *negative-binomial infinitely divisible* (NBID) if for any $p \in (0, 1)$ and $r \in \mathbb{N}$, there is a sequence of i.i.d. random variables $\{Y_{r,p,j}, j \geq 1\}$, such that

$$Y \stackrel{D}{=} \sum_{j=1}^{N_{r,p}} Y_{r,p,j},$$

where $N_{r,p} \sim \text{NB}(r, p)$, independent of Y and $Y_{r,p,j}$ for all $j \geq 1$.

Note that, the NBID random variables defined in Definition 2 are extensions of notion of geometric infinitely divisible (GID) random variables, introduced by Klebanov et al. Obviously, when $r = 1$, the NBID random variable correspond to GID random variables (see [7], Definition 1).

From now on, let us consider a so-called *normalized function* $g : \mathbb{N} \times (0, 1) \mapsto [0, +\infty)$ such that if $r \in \mathbb{N}$ is fixed then $g(r, p) \rightarrow 0$ as $p \rightarrow 0^+$, while if $p \in (0, 1)$ is fixed then $g(r, p) \rightarrow 0$ as $r \rightarrow +\infty$. Since definition of NBID random variables, if there are a normalized function $g(r, p)$ and a sequence of i.i.d. random variables $\{Y_j, j \geq 1\}$ such that

$$Y \stackrel{D}{=} g(r, p) \sum_{j=1}^{N_{r,p}} Y_j,$$

then Y is a NBID random variable, where $N_{r,p} \sim \text{NB}(r, p)$, independent of Y, Y_j for all $j \geq 1$. The following propositions are considered as examples of NBID random variables.

Proposition 3.1. Let $\mathcal{G} \sim \text{Gamma}(r, \lambda r)$ with $\lambda > 0$ and $r \in \mathbb{N}$. Then $\mathcal{G}$ is a NBID random variable. Moreover,

$$\mathcal{G} \xrightarrow{D} \mathcal{D}_{\lambda^{-1}} \quad \text{as } r \rightarrow \infty,$$

where $\mathcal{D}_{\lambda^{-1}}$ is a random variable degenerated at point λ^{-1} .

Proof. We first observe that the first part of Proposition 3.1 is basically the same as Lemma 3.1 of Hung and Hau [27]. Let $\{Z_j, j \geq 1\}$ be a sequence of independent, exponential distributed random variables with parameter $\lambda > 0$, independent of $N_{r,p} \sim \text{NB}(r, p)$. The common characteristic function of $Z_1, Z_2, \dots$ takes the form

$$\varphi_{Z_1}(t) = \frac{\lambda}{\lambda - it}, \quad t \in \mathbb{R}.$$

Since $N_{r,p} \sim \text{NB}(r, p)$, the generating function of $N_{r,p}$ is given by

$$\psi_{N_{r,p}}(w) = \left(\frac{pw}{1 - (1-p)w} \right)^r, \quad \text{for } |w| < \frac{1}{1-p}.$$

Then the characteristic function of negative-binomial sum $\sum_{j=1}^{N_{r,p}} Z_j$ takes the form

$$\varphi_{\sum_{j=1}^{N_{r,p}} Z_j}(t) = \psi_{N_{r,p}}[\varphi_{Z_1}(t)] = \left(\frac{p\lambda}{p\lambda - it} \right)^r, \quad t \in \mathbb{R}.$$

Consider the normalized function $g(r, p) = p/r$. Then, characteristic function of normalized negative-binomial sum $(p/r) \sum_{j=1}^{N_{r,p}} Z_j$ take the form

$$\varphi_{\sum_{j=1}^{N_{r,p}} Z_j}[(p/r)t] = \left(\frac{p\lambda}{p\lambda - i(p/r)t} \right)^r = \left(\frac{\lambda r}{\lambda r - it} \right)^r = \varphi_{\mathcal{G}}(t), \quad t \in \mathbb{R}.$$

Therefore,

$$\mathcal{G} \stackrel{D}{=} (p/r) \sum_{j=1}^{N_{r,p}} Z_j.$$

Moreover, letting $r \rightarrow \infty$, we get

$$\lim_{r \rightarrow \infty} \varphi_{\mathcal{G}}(t) = \lim_{r \rightarrow \infty} \left(\frac{\lambda r}{\lambda r - it} \right)^r = \lim_{r \rightarrow \infty} \left(1 - \frac{it}{\lambda r} \right)^{-r} = \exp \left\{ \frac{1}{\lambda} it \right\} = \varphi_{\mathcal{D}_{\lambda^{-1}}}(t).$$

The proof is straightforward. $\square$

In the sequel, we recall the generalized Linnik laws from [9], page 216 and [5].

Definition 3 ([9] and [5]). A random variable Λ is said to be *generalized Linnik distributed random variable with three parameters* $\alpha \in (0, 2]$, $\sigma > 0$ and $r \in \mathbb{N}$, denoted by $\Lambda \sim \text{GLinnik}(\alpha, \sigma, r)$, if its characteristic function is given by

$$\varphi_{\Lambda}(t) = \left(\frac{1}{1 + \sigma^{\alpha}|t|^{\alpha}} \right)^r, \quad t \in \mathbb{R}.$$

Obviously, generalized Linnik laws correspond to a Linnik laws when $r = 1$. Furthermore, for $\alpha = 2$ and $r = 1$ we obtain a symmetric Laplace distribution (see [9]). Hence, by analogous to Definition 3, we define the generalized Laplace distribution as follows.

Definition 4. A random variable $\mathcal{L}$ is said to be *generalized Laplace distributed random variable with parameters* $\sigma > 0$, and $r \in \mathbb{N}$, denoted by $\mathcal{L} \sim \text{GLaplace}(0, \sigma, r)$, if its characteristic function takes the form

$$\varphi_{\mathcal{L}}(t) = \left(\frac{1}{1 + \frac{\sigma^2}{2}t^2} \right)^r, \quad t \in \mathbb{R}. \quad (6)$$

It is worth mentioning that $\text{GLaplace}(0, \sigma, r)$ is in fact $\text{GLinnik}(2, \frac{\sigma}{\sqrt{2}}, r)$.

Proposition 3.2. Let $\mathcal{L} \sim \text{GLaplace}(0, \sigma r^{-1/2}, r)$ with $r \in \mathbb{N}$ and $\sigma > 0$. Then $\mathcal{L}$ is a NBID random variable. Moreover,

$$\mathcal{L} \xrightarrow{D} \mathcal{N} \quad \text{as } r \rightarrow \infty,$$

where $\mathcal{N} \sim N(0, \sigma)$ is a normal distributed random variable.

Proof. Let $\{L_j, j \geq 1\}$ be a sequence of independent, symmetric Laplace distributed random variables with parameters zero and $\sigma > 0$. The common characteristic function of $L_1, L_2, \dots$ is given by

$$\varphi_{L_1}(t) = \frac{1}{1 + \frac{\sigma^2}{2}t^2}, \quad t \in \mathbb{R}.$$

Let $N_{r,p}$ be a negative-binomial random variable, independent of all $L_1, L_2, \dots$. Then, characteristic function of negative-binomial random sum $\sum_{j=1}^{N_{r,p}} L_j$ is given by

$$\varphi_{\sum_{j=1}^{N_{r,p}} L_j}(t) = \psi_{N_{r,p}}[\varphi_{L_1}(t)] = \left(\frac{p}{p + \frac{\sigma^2}{2} t^2} \right)^r, \quad t \in \mathbb{R}.$$

Thus, characteristic function of normalized negative-binomial sum $(p/r)^{1/2} \sum_{j=1}^{N_{r,p}} L_j$ is defined as follows

$$\varphi_{\sum_{j=1}^{N_{r,p}} L_j}[(p/r)^{1/2} t] = \left(\frac{p}{p + \frac{\sigma^2}{2} (p/r) t^2} \right)^r = \left(\frac{1}{1 + \frac{\sigma^2}{2r} t^2} \right)^r = \varphi_{\mathcal{L}}(t), \quad t \in \mathbb{R}.$$

Therefore,

$$\mathcal{L} \stackrel{D}{=} (p/r)^{1/2} \sum_{j=1}^{N_{r,p}} L_j.$$

Hence, $\mathcal{L}$ is a NBID random variable. Moreover, letting $r \rightarrow \infty$, we get

$$\lim_{r \rightarrow \infty} \varphi_{\mathcal{L}}(t) = \lim_{r \rightarrow \infty} \left(1 + \frac{\sigma^2}{2r} t^2 \right)^{-r} = \exp \left\{ -\frac{\sigma^2}{2} t^2 \right\} = \varphi_{\mathcal{N}}(t).$$

The statement follows by the continuity theorem. $\square$

Proposition 3.3. Let $\Lambda \sim GLinnik(\alpha, \sigma r^{-1/\alpha}, r)$ with $\alpha \in (0, 2]$, $\sigma > 0$ and $r \in \mathbb{N}$. Then Λ is a NBID random variable. Furthermore,

$$\Lambda \xrightarrow{D} \mathcal{S}_{\alpha, \sigma} \quad \text{as } r \rightarrow \infty,$$

where $\mathcal{S}_{\alpha, \sigma} \sim \text{Stable}(\alpha, \sigma)$ is a symmetric stable distributed random variable with characteristic function defined in form

$$\varphi_{\mathcal{S}_{\alpha, \sigma}}(t) = \exp \left\{ -\sigma^\alpha |t|^\alpha \right\}, \quad t \in \mathbb{R}.$$

Proof. Let $\{\xi_j, j \geq 1\}$ be a sequence of independent, symmetric Linnik distributed random variables with parameters $\alpha \in (0, 2]$ and $\sigma > 0$. We will denote by

$$\varphi_{\xi_1}(t) = \frac{1}{1 + \sigma^\alpha |t|^\alpha} \quad \text{for } t \in \mathbb{R},$$

the common characteristic function of $\xi_1, \xi_2, \dots$.

Let $N_{r,p}$ be a negative-binomial random variable, independent of all $\xi_1, \xi_2, \dots$. Then, characteristic function of negative-binomial random sum $\sum_{j=1}^{N_{r,p}} \xi_j$ is defined as follows

$$\varphi_{\sum_{j=1}^{N_{r,p}} \xi_j}(t) = \psi_{N_{r,p}}[\varphi_{\xi_1}(t)] = \left(\frac{p}{p + \sigma^\alpha |t|^\alpha} \right)^r, \quad t \in \mathbb{R}.$$

Thus, characteristic function of a normalized negative-binomial sum $(p/r)^{1/\alpha} \sum_{j=1}^{N_{r,p}} \xi_j$ is given by

$$\varphi_{\sum_{j=1}^{N_{r,p}} \xi_j}[(p/r)^{1/\alpha} t] = \left(\frac{1}{1 + \frac{\sigma^\alpha}{r} |t|^\alpha} \right)^r = \varphi_{\Lambda}(t), \quad t \in \mathbb{R}.$$

Therefore,

$$\Lambda \stackrel{D}{=} (p/r)^{1/\alpha} \sum_{j=1}^{N_{r,p}} \xi_j.$$

Hence, Λ is a NBID random variable. Furthermore, letting $r \rightarrow \infty$, it follows that

$$\lim_{r \rightarrow \infty} \varphi_{\Lambda}(t) = \lim_{r \rightarrow \infty} \left(1 + \frac{\sigma^{\alpha}}{r} |t|^{\alpha}\right)^{-r} = \exp\left\{-\sigma^{\alpha} |t|^{\alpha}\right\} = \varphi_{S_{\alpha, \sigma}}(t), \quad t \in \mathbb{R}.$$

The proof is straightforward. $\square$

4. RATES OF CONVERGENCE IN LIMIT THEOREMS

Using Zolotarev's probability metric, some rates of convergence in limit theorems for normalized negative-binomial sums of i.i.d. random variables will be established in this section. In special cases, several corresponding results for geometric random sums will be indicated.

Theorem 1. *Let $\{X_j, j \geq 1\}$ and $\{Y_j, j \geq 1\}$ be two independent sequences of i.i.d. random variables with $\mathbb{E}|X_1|^s < +\infty$ and $\mathbb{E}|Y_1|^s < +\infty$ for $s \geq 2$. Let $N_{r,p} \sim NB(r, p)$, independent of all X_j and Y_j for $j \geq 1$, $r \in \mathbb{N}$, $p \in (0, 1)$. Let $g(r, p)$ be a normalized function such that (i) If $r \in \mathbb{N}$ be fixed then*

$$\frac{r[g(r, p)]^s}{p} = o(1) \quad \text{as } p \rightarrow 0^+.$$

(ii) If $p \in (0, 1)$ be fixed then

$$\frac{r[g(r, p)]^s}{p} = o(1) \quad \text{as } r \rightarrow \infty.$$

Moreover, assume $C \geq 0$ is a constant such that the following condition holds

$$|\mathbb{E}(X_1^k) - \mathbb{E}(Y_1^k)| \leq C < +\infty \quad \text{for } 1 \leq k \leq m. \quad (7)$$

Then,

$$\zeta_s \left(g(r, p) \sum_{j=1}^{N_{r,p}} X_j, g(r, p) \sum_{j=1}^{N_{r,p}} Y_j; \mathfrak{F}_s \right) \leq \frac{r[g(r, p)]^s}{p} \left\{ \sum_{k=1}^m \frac{C \cdot M_k}{k!} + \frac{1}{m!} (\mathbb{E}|X_1|^s + \mathbb{E}|Y_1|^s) \right\}. \quad (8)$$

Proof. Based on the ideality of Zolotarev's probability metric of order s , we have

$$\begin{aligned} \zeta_s \left(g(r, p) \sum_{j=1}^{N_{r,p}} X_j, g(r, p) \sum_{j=1}^{N_{r,p}} Y_j; \mathfrak{F}_s \right) &= [g(r, p)]^s \zeta_s \left(\sum_{j=1}^{N_{r,p}} X_j, \sum_{j=1}^{N_{r,p}} Y_j; \mathfrak{F}_s \right) \\ &\leq [g(r, p)]^s \sum_{k=1}^{\infty} \left\{ \mathbb{P}(N_{r,p} = k) \zeta_s \left(\sum_{j=1}^k X_j, \sum_{j=1}^k Y_j; \mathfrak{F}_s \right) \right\}. \end{aligned}$$

According to Lemma 1 and assumption (7), from inequality (3), it may be concluded that

$$\begin{aligned} \zeta_s \left(\sum_{j=1}^k X_j, \sum_{j=1}^k Y_j; \mathfrak{F}_s \right) &\leq k \cdot \zeta_s(X_1, Y_1; \mathfrak{F}_s) \\ &\leq k \cdot \left(\sum_{k=1}^m \frac{M_k}{k!} |\mathbb{E}(X_1^k) - \mathbb{E}(Y_1^k)| + \frac{1}{m!} (\mathbb{E}|X_1|^s + \mathbb{E}|Y_1|^s) \right) \\ &\leq k \cdot \left(\sum_{k=1}^m \frac{C \cdot M_k}{k!} + \frac{1}{m!} (\mathbb{E}|X_1|^s + \mathbb{E}|Y_1|^s) \right). \end{aligned}$$

Therefore,

$$\begin{aligned} & \zeta_s \left(g(r, p) \sum_{j=1}^{N_{r,p}} X_j, g(r, p) \sum_{j=1}^{N_{r,p}} Y_j; \mathfrak{F}_s \right) \\ & \leq [g(r, p)]^s \sum_{k=1}^{\infty} \left\{ \mathbb{P}(N_{r,p} = k) k \cdot \left(\sum_{k=1}^m \frac{C \cdot M_k}{k!} + \frac{1}{m!} (\mathbb{E}|X_1|^s + \mathbb{E}|Y_1|^s) \right) \right\} \\ & = \frac{r[g(r, p)]^s}{p} \left\{ \sum_{k=1}^m \frac{C \cdot M_k}{k!} + \frac{1}{m!} (\mathbb{E}|X_1|^s + \mathbb{E}|Y_1|^s) \right\}. \end{aligned}$$

The proof is immediate. $\square$

Let $r \equiv 1$, from Theorem 1, it implies the following result concerning to geometric random sums.

Corollary 1. *Let $\{X_j, j \geq 1\}$ be a sequences of i.i.d. random variables with $\mathbb{E}|X_1|^s < +\infty$ for $s \geq 2$, and Y be a geometrically strictly stable (GSS) random variables. Let $N_{1,p} \sim \text{Geo}(p)$ be a geometric random variable with parameter $p \in (0, 1)$, independent of all X_j and Y for $j \geq 1$. Assume that $C \geq 0$, the following condition holds*

$$|\mathbb{E}(X_1^k) - \mathbb{E}(Y^k)| \leq C < +\infty \quad \text{for } 1 \leq k \leq m.$$

Then,

$$\zeta_s \left(c(p) \sum_{j=1}^{N_{1,p}} X_j, Y; \mathfrak{F}_s \right) \leq \frac{[c(p)]^s}{p} \left\{ \sum_{k=1}^m \frac{C \cdot M_k}{k!} + \frac{1}{m!} (\mathbb{E}|X_1|^s + \mathbb{E}|Y_1|^s) \right\}.$$

where $c(p) > 0$ be a constant such that $\frac{[c(p)]^s}{p} = o(1)$ as $p \rightarrow 0^+$.

Proof. Since Y be a geometrically strictly stable random variables, according to ([7], Definition 2, page 793), we get the following presentation

$$Y \stackrel{D}{=} c(p) \sum_{j=1}^{N_{1,p}} Y_j,$$

where $Y_1, Y_2, \dots$ are copies of Y and they are independent of $N_{1,p}$. Applying to Theorem 1 for $r = 1$ and $g(1, p) = c(p)$ then the proof is finished. $\square$

Theorem 2. *Let $\{X_j, j \geq 1\}$ be a sequence of positive i.i.d. random variables with $\mathbb{E}(X_1) = \mu \in (0, +\infty)$ and $\mathbb{E}(X_1^2) = \tau^2 \in (0, +\infty)$. Let $N_{r,p} \sim \text{NB}(r, p)$, independent of X_j for all $j \geq 1$. Let $\mathcal{G} \sim \text{Gamma}(r, \mu^{-1}r)$ be a Gamma random variable with parameters $r \in \mathbb{N}$ and $\mu^{-1}r$. Then,*

$$\zeta_2 \left((p/r) \sum_{j=1}^{N_{r,p}} X_j, \mathcal{G}; \mathfrak{F}_2 \right) \leq \frac{p}{r} \left(\tau^2 + \frac{2}{\mu^2} \right).$$

Moreover,

(i) *Let $r \in \mathbb{N}$ be a fixed number. Then*

$$(p/r) \sum_{j=1}^{N_{r,p}} X_j \xrightarrow{D} \mathcal{G} \quad \text{as } p \rightarrow 0^+. \quad (9)$$

(ii) *Let $p \in (0, 1)$ be a fixed number. Then*

$$(p/r) \sum_{j=1}^{N_{r,p}} X_j \xrightarrow{D} \mathcal{D}_\mu \quad \text{as } r \rightarrow \infty, \quad (10)$$

where $\mathcal{D}_\mu$ is a random variable degenerated at point μ , whose characteristic function is given by

$$\varphi_{\mathcal{D}_\mu}(t) = e^{it\mu}, \quad t \in \mathbb{R}.$$

Proof. According to Proposition 3.1, $\mathcal{G}$ will be a NBID random variable. Hence,

$$\mathcal{G} \stackrel{D}{=} (p/r) \sum_{j=1}^{N_{r,p}} Z_j,$$

where $\{Z_j, j \geq 1\}$ be a sequence of independent exponential distributed random variables with parameter μ^{-1} and they are independent of $N_{r,p}$. Moreover,

$$\mathbb{E}(Z_1) = \mu = \mathbb{E}(X_1) \quad \text{and} \quad \mathbb{E}(Z_1^2) = \frac{2}{\mu^2}.$$

From Theorem 1 with $m = 1$, $s = 2$ and normalized function $g(r, p) = p/r$, we conclude that

$$\zeta_2 \left((p/r) \sum_{j=1}^{N_{r,p}} X_j, \mathcal{G}; \mathfrak{F}_2 \right) \leq \frac{r(p/r)^2}{p \cdot 1!} (\mathbb{E}|X_1|^2 + \mathbb{E}|Z_1|^2) = \frac{p}{r} \left(\tau^2 + \frac{2}{\mu^2} \right).$$

(i) According to Remark 2.1, we get the confirmation (9).

(ii) By the triangle inequality of Zolotarev's probability metric, we have

$$\zeta_2 \left((p/r) \sum_{j=1}^{N_{r,p}} X_j, \mathcal{D}_\mu; \mathfrak{F}_2 \right) \leq \zeta_2 \left((p/r) \sum_{j=1}^{N_{r,p}} X_j, \mathcal{G}; \mathfrak{F}_2 \right) + \zeta_2(\mathcal{G}, \mathcal{D}_\mu; \mathfrak{F}_2).$$

Thanks to Remark 2.1 and Proposition 3.1, we get

$$\begin{aligned} \zeta_2 \left((p/r) \sum_{j=1}^{N_{r,p}} X_j, \mathcal{G}; \mathfrak{F}_2 \right) &\longrightarrow 0 \quad \text{as } r \rightarrow \infty; \\ \zeta_2(\mathcal{G}, \mathcal{D}_\mu; \mathfrak{F}_2) &\longrightarrow 0 \quad \text{as } r \rightarrow \infty. \end{aligned}$$

Therefore, the statement (10) has been established. The proof is finished. $\square$

Next Rényi-type theorem will be concluded from Theorem 2 as a direct consequence.

Corollary 2 (Rényi-type theorem). Let $\{X_j, j \geq 1\}$ be a sequence of positive i.i.d. random variables with $\mathbb{E}(X_1) = \mu \in (0, +\infty)$ and $\mathbb{E}(X_1^2) = \tau^2 \in (0, +\infty)$. Let $N_{1,p} \sim \text{Geo}(p)$ be a geometric random variable with parameter $p \in (0, 1)$ and independent of X_j for all $j \geq 1$. Then,

$$\zeta_2 \left(p \sum_{j=1}^{N_{1,p}} X_j, Z; \mathfrak{F}_2 \right) \leq p \left(\tau^2 + \frac{2}{\mu^2} \right),$$

where $Z \sim \text{Exp}(\mu^{-1})$. Moreover,

$$p \sum_{j=1}^{N_{1,p}} X_j \xrightarrow{D} Z \quad \text{as } p \rightarrow 0^+.$$

Proof. The proof is immediate from Theorem 2 when $r = 1$. $\square$

Theorem 3. Let $\{X_j, j \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}(X_1) = 0$, $\mathbb{E}(X_1^2) = \sigma^2 \in (0, +\infty)$ and $\mathbb{E}|X_1|^3 = \vartheta \in (0, +\infty)$. Let $N_{r,p} \sim \text{NB}(r, p)$, independent of all X_j , $j \geq 1$. Assume that $\mathcal{L} \sim \text{GLaplace}(0, \sigma r^{-1/2}, r)$ with $\sigma > 0$. Then,

$$\zeta_3 \left((p/r)^{1/2} \sum_{j=1}^{N_{r,p}} X_j, \mathcal{L}; \mathfrak{F}_3 \right) \leq (p/r)^{1/2} \left(\frac{\vartheta}{2} + \frac{3\sigma^3}{2\sqrt{2}} \right).$$

Moreover,

(i) Let $r \in \mathbb{N}$ be a fixed number. Then

$$(p/r)^{1/2} \sum_{j=1}^{N_{r,p}} X_j \xrightarrow{D} \mathcal{L} \quad \text{as } p \rightarrow 0^+. \quad (11)$$

(ii) Let $p \in (0, 1)$ be a fixed number. Then

$$(p/r)^{1/2} \sum_{j=1}^{N_{r,p}} X_j \xrightarrow{D} \mathcal{N} \sim N(0, \sigma) \quad \text{as } r \rightarrow \infty. \quad (12)$$

Proof. According to Proposition 3.2, $\mathcal{L} \sim \text{GLaplace}(0, \sigma r^{-1/2}, r)$ will be a NBID random variable and

$$\mathcal{L} \stackrel{D}{=} (p/r)^{1/2} \sum_{j=1}^{N_{r,p}} L_j,$$

where $\{L_j, j \geq 1\}$ be a sequence of independent symmetric Laplace distributed random variables with parameters zero and $\sigma > 0$. Note that $N_{r,p}$ is independent of all L_j for $j \geq 1$. Moreover,

$$\mathbb{E}(L_1) = 0 = \mathbb{E}(X_1); \quad \mathbb{E}(L_1^2) = \sigma^2 = \mathbb{E}(X_1^2) \quad \text{and} \quad \mathbb{E}|L_1|^3 = \frac{3\sigma^3}{\sqrt{2}}.$$

On account of Theorem 1 with a normalized function $g(r, p) = (p/r)^{1/2}$, $m = 2$ and $s = 3$, it follows that

$$\zeta_3 \left((p/r)^{1/2} \sum_{j=1}^{N_{r,p}} X_j, \mathcal{L}; \mathfrak{F}_3 \right) \leq \frac{r(p/r)^{3/2}}{p \cdot 2!} (\mathbb{E}|X_1|^3 + \mathbb{E}|L_1|^3) = (p/r)^{1/2} \left(\frac{\vartheta}{2} + \frac{3\sigma^3}{2\sqrt{2}} \right).$$

(i) According to Remark 2.1 the confirmation (11) will be obtained immediately.

(ii) By the triangle inequality of Zolotarev's probability metric, we have

$$\zeta_3 \left((p/r)^{1/2} \sum_{j=1}^{N_{r,p}} X_j, \mathcal{N}; \mathfrak{F}_3 \right) \leq \zeta_3 \left((p/r)^{1/2} \sum_{j=1}^{N_{r,p}} X_j, \mathcal{L}; \mathfrak{F}_3 \right) + \zeta_3(\mathcal{L}, \mathcal{N}; \mathfrak{F}_3).$$

Therefore, combining with Remark 2.1 and Proposition 3.2, one has

$$\begin{aligned} \zeta_3 \left((p/r)^{1/2} \sum_{j=1}^{N_{r,p}} X_j, \mathcal{L}; \mathfrak{F}_3 \right) &\longrightarrow 0 \quad \text{as } r \rightarrow \infty; \\ \zeta_3(\mathcal{L}, \mathcal{N}; \mathfrak{F}_3) &\longrightarrow 0 \quad \text{as } r \rightarrow \infty. \end{aligned}$$

The (12) will be concluded. The proof is complete. $\square$

Corollary 3. Let $\{X_j, j \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}(X_1) = 0$, $\mathbb{E}(X_1^2) = \sigma^2 \in (0, +\infty)$ and $\mathbb{E}|X_1|^3 = \vartheta \in (0, +\infty)$. Let $N_{1,p} \sim \text{Geo}(p)$ be a geometric random variable with parameter $p \in (0, 1)$ and independent of all X_j for $j \geq 1$. Then

$$\zeta_3 \left(p^{1/2} \sum_{j=1}^{N_{1,p}} X_j, L; \mathfrak{F}_3 \right) \leq p^{1/2} \left(\frac{\vartheta}{2} + \frac{3\sigma^3}{2\sqrt{2}} \right),$$

where $L \sim \text{Laplace}(0, \sigma)$ with $\sigma > 0$. Moreover,

$$p^{1/2} \sum_{j=1}^{N_{1,p}} X_j \xrightarrow{D} L \sim \text{Laplace}(0, \sigma) \quad \text{as } p \rightarrow 0^+.$$

Proof. The proof is immediate from Theorem 3 when $r \equiv 1$. $\square$

It is worth pointing out that Theorem 1 could not be applied for a normalized negative-binomial random sum $(p/r)^{1/\alpha} S_{N_{r,p}}$ for $\alpha \in (0, 2]$. The main reason is that the limiting distributions of $(p/r)^{1/\alpha} S_{N_{r,p}}$ as $p \rightarrow 0^+$ concerning to Linnik laws, whose absolute moments are finite for $0 < s < \alpha$ while they are infinite for $s \geq \alpha$ (see [9], page 212).

Theorem 4. Let $\{X_j, j \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}(X_1) = 0$ and $\mathbb{E}|X_1| = \varrho \in (0, +\infty)$. Let $N_{r,p} \sim NB(r, p)$, independent of all $X_j, j \geq 1$. Assume that $\Lambda \sim GLinnik(\alpha, \sigma r^{-1/\alpha}, r)$ with $\alpha \in (1, 2)$ and $\sigma > 0$. Then

$$\zeta_2 \left((p/r)^{1/\alpha} \sum_{j=1}^{N_{r,p}} X_j, \Lambda; \mathfrak{F}_2 \right) \leq 2M(p/r)^{\frac{2-\alpha}{\alpha}} \left(\varrho + \frac{2\sigma}{\alpha \sin(\frac{\pi}{\alpha})} \right), \quad (13)$$

where $\|f'\| = \sup_{w \in \mathbb{R}} |f'(w)|$ and $M = \sup_{f \in \mathfrak{F}_2} \|f'\|$.

Moreover,

(i) Let $r \in \mathbb{N}$ be a fixed number. Then

$$(p/r)^{1/\alpha} \sum_{j=1}^{N_{r,p}} X_j \xrightarrow{D} \Lambda \quad \text{as } p \rightarrow 0^+. \quad (14)$$

(ii) Let $p \in (0, 1)$ be a fixed number. Then

$$(p/r)^{1/\alpha} \sum_{j=1}^{N_{r,p}} X_j \xrightarrow{D} \mathcal{S}_{\alpha, \sigma} \quad \text{as } r \rightarrow \infty, \quad (15)$$

where $\mathcal{S}_{\alpha, \sigma} \sim \text{Stable}(\alpha, \sigma)$.

Proof. Let $\{\xi_j, j \geq 1\}$ be a sequence of independent, symmetric Linnik distributed random variables with parameters $\alpha \in (1, 2)$, $\sigma > 0$. Assume that all $\xi_j, j \geq 1$ are independent of $N_{r,p}$. On account of Proposition 3.3, $\Lambda \sim GLinnik(\alpha, \sigma r^{-1/\alpha}, r)$ will be a NBID random variable and

$$\Lambda \stackrel{D}{=} (p/r)^{1/\alpha} \sum_{j=1}^{N_{r,p}} \xi_j.$$

Based on the ideality of Zolotarev's probability metric of order $s = 2$ and according to Remark 2.1, it follows that

$$\begin{aligned} \zeta_2 \left((p/r)^{1/\alpha} \sum_{j=1}^{N_{r,p}} X_j, \Lambda; \mathfrak{F}_2 \right) &= \zeta_2 \left((p/r)^{1/\alpha} \sum_{j=1}^{N_{r,p}} X_j, (p/r)^{1/\alpha} \sum_{j=1}^{N_{r,p}} \xi_j; \mathfrak{F}_2 \right) \\ &= (p/r)^{2/\alpha} \zeta_2 \left(\sum_{j=1}^{N_{r,p}} X_j, \sum_{j=1}^{N_{r,p}} \xi_j; \mathfrak{F}_2 \right) \\ &= (p/r)^{2/\alpha} \sum_{k=1}^{\infty} \left\{ \mathbb{P}(N_{r,p} = k) \cdot \zeta_2 \left(\sum_{j=1}^k X_j, \sum_{j=1}^k \xi_j; \mathfrak{F}_2 \right) \right\} \\ &\leq (p/r)^{2/\alpha} \sum_{k=1}^{\infty} \left\{ \mathbb{P}(N_{r,p} = k) k \cdot \zeta_2(X_1, \xi_1; \mathfrak{F}_2) \right\} \\ &= (p/r)^{2/\alpha} \mathbb{E}(N_{r,p}) \cdot \zeta_2(X_1, \xi_1; \mathfrak{F}_2) \\ &= (p/r)^{\frac{2-\alpha}{\alpha}} \cdot \zeta_2(X_1, \xi_1; \mathfrak{F}_2). \end{aligned}$$

For any $f \in \mathfrak{F}_2 \subset \mathfrak{D}_2$, and for any $w \in \mathbb{R}$, we have

$$|f'(w) - f'(0)| \leq 2\|f'\|.$$

By the Mean Value Theorem (see [19], page 107), for w comes between x and y , it follows that

$$f(x) - f(y) = (x - y)f'(w) = (x - y)f'(0) + (x - y)[f'(w) - f'(0)].$$

Thus, we infer that

$$f(x) - f(y) \leq (x - y)f'(0) + 2\|f'\|(|x| + |y|).$$

According to Kotz et al. (2001) (see [9], page 212), since $\xi_1 \sim \text{Linnik}(\alpha, \sigma)$ with $\alpha \in (1, 2)$ and $\sigma > 0$, we have

$$\mathbb{E}(\xi_1) = 0 \quad \text{and} \quad \mathbb{E}|\xi_1| = \frac{2\sigma}{\alpha \sin(\frac{\pi}{\alpha})} < +\infty.$$

Using assumptions that $\mathbb{E}(X_1) = 0$ and $\mathbb{E}|X_1| = \varrho \in (0, \infty)$, it follows that

$$\begin{aligned} \mathbb{E}[f(X_1) - f(\xi_1)] &\leq \mathbb{E}\left[(X_1 - \xi_1)f'(0) + 2\|f'\|(|X_1| + |\xi_1|)\right] \\ &= 2\|f'\| \left(\varrho + \frac{2\sigma}{\alpha \sin(\frac{\pi}{\alpha})}\right). \end{aligned}$$

Letting supremum with $f \in \mathfrak{F}_2$ and let us set $M = \sup_{f \in \mathfrak{F}_2} \|f'\|$, it may be conclude that

$$\zeta_2(X_1, \xi_1) \leq 2M \left(\varrho + \frac{2\sigma}{\alpha \sin(\frac{\pi}{\alpha})}\right).$$

Finally, we obtain

$$\zeta_2 \left((p/r)^{1/\alpha} \sum_{j=1}^{N_{r,p}} X_j, \Lambda; \mathfrak{F}_2 \right) \leq (p/r)^{\frac{2-\alpha}{\alpha}} 2M \left(\varrho + \frac{2\sigma}{\alpha \sin(\frac{\pi}{\alpha})}\right).$$

Moreover,

- (i) From (13), Remark 2.1 and based on the finiteness of σ , ϱ and M , the (14) will be established.
- (ii) By an analogous to previous theorems, using the triangle inequality we get

$$\zeta_2 \left((p/r)^{1/\alpha} \sum_{j=1}^{N_{r,p}} X_j, \mathcal{S}_{\alpha,\sigma}; \mathfrak{F}_2 \right) \leq \zeta_2 \left((p/r)^{1/\alpha} \sum_{j=1}^{N_{r,p}} X_j, \Lambda; \mathfrak{F}_2 \right) + \zeta_2(\Lambda, \mathcal{S}_{\alpha,\sigma}; \mathfrak{F}_2).$$

Since the estimation (13) and Proposition 3.3, for p fixed, one has

$$\begin{aligned} \zeta_2 \left((p/r)^{1/\alpha} \sum_{j=1}^{N_{r,p}} X_j, \Lambda; \mathfrak{F}_2 \right) &\longrightarrow 0 \quad \text{as } r \rightarrow \infty; \\ \zeta_2(\Lambda, \mathcal{S}_{\alpha,\sigma}; \mathfrak{F}_2) &\longrightarrow 0 \quad \text{as } r \rightarrow \infty. \end{aligned}$$

Therefore, the (15) will be concluded. The proof is straightforward. $\square$

Corollary 4. *Let $\{X_j, j \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}(X_1) = 0$ and $\mathbb{E}|X_1| = \varrho \in (0, +\infty)$. Let $N_{1,p} \sim \text{Geo}(p)$, independent of all $X_j, j \geq 1$. Moreover, for any $\alpha \in (1, 2)$ and $\sigma > 0$, assume that the random variable $\xi \sim \text{Linnik}(\alpha, \sigma)$. Then,*

$$\zeta_2 \left(p^{1/\alpha} \sum_{j=1}^{N_{1,p}} X_j, \xi; \mathfrak{F}_2 \right) \leq 2Mp^{\frac{2-\alpha}{\alpha}} \left(\varrho + \frac{2\sigma}{\alpha \sin(\frac{\pi}{\alpha})}\right),$$

where $\|f'\| = \sup_{w \in \mathbb{R}} |f'(w)|$ and $M = \sup_{f \in \mathfrak{F}_2} \|f'\|$. Furthermore, a limit theorem for normalized geometric random sums will be formulated as follows

$$p^{1/\alpha} \sum_{j=1}^{N_{1,p}} X_j \xrightarrow{D} \xi \sim \text{Linnik}(\alpha, \sigma) \quad \text{as } p \rightarrow 0^+.$$

Proof. The proof is straightforward from Theorem 4 when $r \equiv 1$. $\square$

5. CONCLUDING REMARKS

Let $\{X_j, j \geq 1\}$ be a sequence of i.i.d. random variables, $N_{1,p} \sim \text{Geo}(p)$ and $N_{r,p} \sim \text{NB}(r, p)$. Assume that all X_j independent of $N_{1,p}$ and $N_{r,p}$ for all $j \geq 1$. Consider three following sums

$$S_n = \sum_{j=1}^n X_j, \quad S_{N_{1,p}} = \sum_{j=1}^{N_{1,p}} X_j \quad \text{and} \quad S_{N_{r,p}} = \sum_{j=1}^{N_{r,p}} X_j.$$

The first sum is a *deterministic sum*, the second sum is a *geometric random sum* and the third sum is so-called *negative-binomial random sum*. Based on received results, we conclude this paper with the following comments

1. The limiting distributions of the normalized deterministic sums S_n and the normalized geometric random sums $S_{N_{1,p}}$ should be shown in following table.

The deterministic sums		The geometric random sums	
Nor. Sums	Limited dist. as $n \rightarrow \infty$	Nor. Geo. Sums	Limited dist. as $p \rightarrow 0^+$
$n^{-1} S_n$	Degenerate dist.	$p S_{N_{1,p}}$	Exponential dist.
$n^{-1/2} S_n$	Normal dist.	$p^{1/2} S_{N_{1,p}}$	Laplace dist.
$n^{-1/\alpha} S_n$	Stable dist.	$p^{1/\alpha} S_{N_{1,p}}$	Linnik dist.

2. The comments on limiting distributions of various normalized negative-binomial random sums will be presented as follows.

Nor. Negative-binomial Sums	Limited dist. as $r \rightarrow \infty$	Limited dist. as $p \rightarrow 0^+$
$(p/r) S_{N_{r,p}}$	Degenerate dist.	Gamma dist.
$(p/r)^{1/2} S_{N_{r,p}}$	Normal dist.	Extended Laplace dist.
$(p/r)^{1/\alpha} S_{N_{r,p}}$	Stable dist.	Extended Linnik dist.

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